Some Applications On Inclusion Relations for Uniformly Starlike and Convex Functions

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A B S T R A C T

In this paper, we have established the inclusion relations for k-uniformly starlike functions under the multiplier transformation I_n^{λ} operator. These results are also extended to k- uniformly convex functions,

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1. Introduction.

Put ∇ in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

And $D = \{z \in \mathbb{C};\ |z| < 1\}$.

If f and g are holomorphic in D, we called $°\mathrm{F}$ is subordinate to ∂ , written $°\mathrm{F} \prec \partial$ or $°\mathrm{F}(z) \prec \partial(z)$,

if there a Schwartz function $\propto$ in D such that $°\mathrm{F}(z) = \partial(\propto(z))$.

For any $0 \leq \gamma < 1,\ k \geq 0$, we define $UST\ (k,\gamma), UCV\ (k,\gamma), UCC\ (k,\gamma,\beta)$ and $UQC\ (k,\gamma,\beta)$ the g-uniformly subclasses as follows :

$$UST\ (k,\gamma) = \left\{ f \in \nabla : \Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \right\} \qquad (2)$$

$$UCV\ (k,\gamma) = \left\{ f \in \nabla : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \right\} \qquad (3)$$

$$UCC\ (k,\gamma,\beta) = \left\{ f \in \nabla : \exists\ g \in UST\ (k,\gamma), \Re\left(\frac{zf'(z)}{g(z)} - \gamma\right) \geq k\left|\frac{zf'(z)}{g(z)} - 1\right| \right\} \qquad (4)$$

$$UQC\ (k,\gamma,\beta) = \left\{ f \in \nabla : \exists\ g \in UCV\ (k,\gamma), \Re\left(\frac{(zf'(z))'}{g'(z)} - \gamma\right) \geq k\left|\frac{zf'(z)}{g'(z)} - 1\right| \right\} \qquad (5).$$

For more details see [1].

Setting

$$\Omega_{k,\gamma} = \left\{u + iv, u > k\sqrt{(u-1)^2 + v^2} + \gamma\right\} \qquad (6)$$

where $\vartheta(z) = \frac{zf'(z)}{f(z)}$ or $\vartheta(z) = 1 + \frac{zf''(z)}{f'(z)}$ and considering the functions which maps D on to the conic domain $\Omega_{k,\gamma}$, such that $1 \in \Omega_{k,\gamma}$, we may rewrite the conditions (2) or (3) in the form $p(z) \prec q_{k,\gamma}$. We introduce the function $q_{k,\gamma}$ as the following :

We note that $I_n^\lambda\left(I_m^\lambda f(z)\right) = I_{n+m}^\lambda f(z)$ for all integers m and n . The operators I_n^λ are closely related to the Komatu integral operators [8] and the differential and integral operators defined by Salagean[2]. We also note that $I_0^0 f(z) = zf'(z)$ and $I_0^1 f(z) = f(z)$ the operator defined by Cho and Kim[5].

Next, using the operator I_n^λ, we introduce the following k-uniformaly class of functions for $0 \leq \gamma < 1$, $k \geq 0$ and $0 \leq \lambda$:

$$UST\ (\lambda, k, \gamma) = \left\{f \in A : I_n^\lambda f(z) \in UST\ (k, \gamma); z \in D\right\} \tag{11}$$

$$UCV\ (\lambda, k, \gamma) = \left\{f \in A : I_\lambda^\lambda f(z) \in UCV\ (k, \gamma); z \in D\right\} \tag{12}$$

$$UCC\ (\lambda, k, \gamma, \beta) = \left\{f \in A : I_n^\lambda f(z) \in UCC\ (k, \gamma, \beta); z \in D\right\} \tag{13}$$

$$UQC\ (\lambda, k, \gamma, \beta) = \{f \in A :\} I_n^\lambda f(z) \in UQC\ (k, \gamma, \beta); z \in D \qquad (14).$$

We also note that

$$f \in UST\ (\lambda, k, \gamma) \leftrightarrow zf' \in UCV\ (\lambda, k, \gamma), \tag{15}$$

$$f \in\ UCC\ (\lambda, k, \gamma, \beta) \leftrightarrow zf' \in UQC\ (\lambda, k, \gamma, \beta). \tag{16}$$

1. Inclusion properties

In the study we shall to prove some theorems involving operator I_{n+1}^λ .
to prove main theorems, we shall need the following lemmas.

"Lemma 1[6]: Let β and γ is complex, h is univalent convex in the open disk D, where $h(0) = c$ and $\Re(\beta h(z) + \gamma) > 0$. Let $g(z) = c + \sum_{n=1}^{\infty} b_n z^n$ be holomorphic in D. Then

$$g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} \prec h(z) \rightarrow g(z) \prec h(z)."$$

"Lemma 2[7]: Let h is convex in the open disk D and let $A \geq 0$. set $B(z)$ is holomorphic in D with $\Re(B(z)) \geq A$. If g is holomorphic in D and $g(0) = h(0)$. Then

$$Az^2 g''(z) + B(z)zg'(z) + g(z) \prec h(z) \rightarrow g(z) \prec h(z)."$$

Theorem 1. Let $f(z) \in \nabla$. If $I_n^\lambda f(z) \in UST(k, \gamma)$, then $I_{n+1}^\lambda f(z) \in UST(k, \gamma)$.
Proof. Let $I_n^\lambda f(z) \in UST(k, \gamma)$ and set

$$\vartheta(z) = \frac{z\left(I_{n+1}^\lambda f(z)\right)'}{I_{n+1}^\lambda f(z)} . \tag{17}$$

Using (10) and (17), we have

$$\frac{I_n^\lambda f(z)}{I_{n+1}^\lambda f(z)} = \frac{\vartheta(z) + \lambda}{\lambda + 1}. \tag{18}$$

Differentiating the last equation yields

$$\frac{z\left(I_{n+1}^\lambda f(z)\right)'}{I_{n+1}^\lambda f(z)} = \vartheta(z) + \frac{z\vartheta'(z)}{\vartheta(z) + \lambda} , \tag{19}$$

from (8) we have

$$\vartheta(z) + \frac{z\vartheta'(z)}{\vartheta(z) + \lambda} \prec q_{k,\gamma}\ (z).$$

By using (1) to (19) and Lemma1, we get $\vartheta(z) \prec q_{k,\gamma}\ (z)$, hence

$$I_{n+1}^\lambda f(z) \in UST(k, \gamma).$$

Theorem 2. Let $f(z) \in \nabla$. If $I_n^\lambda f(z) \in UCV(k, \gamma)$, then $I_{n+1}^\lambda f(z) \in UCV(k, \gamma)$.

$$I_n^\lambda f(z) \in UST(k,\gamma) \leftrightarrow z\left(I_n^\lambda f(z)\right)' \in UST(k,\gamma)$$
$$\leftrightarrow I_n^\lambda zf'(z) \in UST(k,\gamma)$$
$$\to I_{n+1}^\lambda zf'(z) \in UST(k,\gamma)$$
$$\leftrightarrow I_{n+1}^\lambda f(z) \in UCV(k,\gamma).$$

and the proof of Theorem 2 is completed.

Theorem 3. Let $f(z) \in \nabla$. If $I_n^\lambda f(z) \in UCC(k,\gamma,\beta)$, then $I_{n+1}^\lambda f(z) \in UCC(k,\gamma,\beta)$.

Proof. Since $I_n^\lambda f(z) \in UCC(k,\gamma,\beta)$ by definition we can write

$$\frac{z\left(I_n^\lambda f(z)\right)'}{k(z)} \prec q_{k,\gamma}(z)$$

for some $k(z) \in UST(k,\gamma)$. For $g(z)$ such that $I_n^\lambda g(z) = k(z)$ we have

$$\frac{z\left(I_n^\lambda f(z)\right)'}{I_n^\lambda g(z)} \prec q_{k,\gamma}(z). \tag{20}$$

Let
$$h(z) = \frac{z\left(I_{n+1}^\lambda f(z)\right)'}{I_{n+1}^\lambda g(z)}$$

and
$$H(z) = \frac{z\left(I_{n+1}^\lambda g(z)\right)'}{I_{n+1}^\lambda g(z)}$$

where $h(z)$ and $H(z)$ are analytic in U and $h(0) = (0) = 1$.

Now, by Theorem 1

$$I_{n+1}^\lambda g(z) \in UST(k,\gamma) \text{ and } \Re(H(z)) > \frac{k+\gamma}{k+1}.$$

Also, we note that

$$z\left(I_{n+1}^\lambda f(z)\right)' = \left(I_{n+1}^\lambda g(z)\right)h(z). \tag{21}$$

Differentiating both sides of (21) we have

$$\frac{z\left(I_{n+1}^\lambda (zf'(z))\right)'}{I_{n+1}^\lambda g(z)} = \frac{z\left(I_{n+1}^\lambda g(z)\right)'}{I_{n+1}^\lambda g(z)} \cdot h(z) + zh'(z)$$
$$= H(z)h(z) + zh'(z).$$

Now using the identity (10), we obtain

$$\frac{z\left(I_n^\lambda f(z)\right)'}{I_n^\lambda g(z)} = \frac{I_n^\lambda (zf'(z))}{I_n^\lambda g(z)} = \frac{z(I_{n+1}^\lambda (zf'(z)))' + \lambda I_{n+1}^\lambda (zf'(z))}{z\left(I_{n+1}^\lambda g(z)\right)' + \lambda I_{n+1}^\lambda g(z)}$$

$$= \frac{\frac{z(I_{n+1}^\lambda (zf'(z)))'}{I_{n+1}^\lambda g(z)} + \frac{\lambda I_{n+1}^\lambda (zf'(z))}{I_{n+1}^\lambda g(z)}}{\frac{z\left(I_{n+1}^\lambda g(z)\right)'}{I_{n+1}^\lambda g(z)} + \lambda}$$
$$= \frac{H(z)h(z) + zh'(z) + h(z)}{H(z) + \lambda}$$
$$= h(z) + \frac{zh'(z)}{H(z)+\lambda} \tag{22}$$

from (20),(21) and (22), we conclude that

$$h(z) + \frac{zh'(z)}{H(z) + \lambda} \prec q_{k,\gamma}(z).$$

Let $A = 0$ and $B(z) = \frac{1}{H(z)+\lambda}$ we have

$$\Re(B(z)) = \frac{1}{|H(z)+\lambda|^2}\,\Re(\lambda + H(z)) > 0.$$

This inequality satisfies conditions in Lemma 2. Since
$h(z) \prec q_{k,\gamma}(z)$, completed the proof.

Theorem 4. put $f(z) \in \nabla$. If $I_n^\lambda f(z) \in UQC(k,\gamma,\beta)$, then $I_{n+1}^\lambda f(z) \in UQC(k,\gamma,\beta)$.
Proof. The proof of Theorem 4 similar that Theorem 3.

3. the closure properties of the operator F_C.

Now, we consider $F_c(z)$ [4],[3], which defined by

$$F_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt\,, \qquad (c \geq 0). \tag{23}$$

Theorem 5 . Set $c > \frac{-(k+\gamma)}{k+1}$.
If $I_n^\lambda f(z) \in UST(k,\gamma)$, then $I_n^\lambda F_C(f)(z) \in UST(k,\gamma)$.
Proof. Let $f(z) \in UST(k,\gamma)$ and set

$$\vartheta(z) = \frac{z\left(I_n^\lambda F_c(f)(z)\right)'}{I_n^\lambda F_c(f)(z)}$$

where, $p(z)$ analytic in D. From (23), we get

$$z\left(I_n^\lambda F_c(f)(z)\right)' = (c+1)I_n^\lambda f(z) - cI_n^\lambda F_c(f)(z). \tag{24}$$

Then, by using (24) and we obtain

$$\vartheta(z) + c = (c+1)\frac{I_n^\lambda f(z)}{I_n^\lambda F_c(f)(z)}\,. \tag{25}$$

By differentiation equatin (25) we, have

$$\frac{z\left(I_n^\lambda f(z)\right)'}{I_n^\lambda f(z)} = \vartheta(z) + \frac{z\vartheta'(z)}{\vartheta(z)+c} \prec q_{k,\gamma}(z). \tag{26}$$

Hence, by virtue of Lemma 1, we conclude that $\vartheta(z) \prec q_{k,\gamma}(z)$, which implies that
$I_n^\lambda F_C(f)(z) \in UST(k,\gamma)$.

Theorem 6. Suppose $c > \frac{-(k+\gamma)}{k+1}$.
If $I_n^\lambda f(z) \in UCV(k,\gamma)$, then $I_n^\lambda F_C(f)(z) \in UCV(k,\gamma)$.
Proof. By applying (15) and Theorem 5, it follows that

$$\begin{aligned} f(z) \in UST(k,\gamma) \quad &\leftrightarrow zf'(z) \in UST(k,\gamma) \\ &\rightarrow F_c(zf'(z)) \in UST(k,\gamma) \quad \text{by Theorem 5} \\ &\leftrightarrow z(F_c(f)(z))' \in UST(k,\gamma) \\ &\leftrightarrow F_c(f)(z) \in UCV(k,\gamma) \end{aligned}$$

which proves Theorem 6.

Theorem 7. Suppose $c > \frac{-(k+\gamma)}{k+1}$.
If $I_n^\lambda f(z) \in UCC(k,\gamma,\beta)$, then $I_n^\lambda F_C(f)(z) \in UCC(k,\gamma,\beta)$.
Proof. Let $f(z) \in UCC(k,\gamma,\beta)$. Then by the definition of the class $UCC(k,\gamma,\beta)$ there exists a function $g(z) \in UST(k,\gamma)$ such that

$$\frac{z\left(I_n^\lambda f(z)\right)'}{I_n^\lambda g(z)} \prec q_{k,\gamma}\ (z).$$

Thus we set

$$h(z) = \frac{z\left(I_n^\lambda F_c(f)(z)\right)'}{I_n^\lambda F_c(f)(z)} \tag{27}$$

where $h(z)$ analytic in D. Since $g(z) \in UST(k,\gamma)$ we see from Theorem 5 that $F_c(g)(z) \in UST(k,\gamma)$. Let

$$H(z) = \frac{z\left(I_n^\lambda F_c(g)(z)\right)'}{I_n^\lambda F_c(g)(z)} \tag{28}$$

where $H(z)$ holomorphic in D where, $\Re(H(z)) > \frac{k+\gamma}{k+1}$, from (27) we conclude

$$z\left(I_n^\lambda F_c(f)(z)\right)' = h(z)\left(I_n^\lambda F_c(g)(z)\right). \tag{29}$$

Differentiating both sides of (29) we obtain

$$\begin{aligned} \frac{z\left(I_n^\lambda (zF_c'(f)(z))\right)'}{I_n^\lambda F_c(g)(z)} &= \frac{z\left(I_n^\lambda F_c(g)(z)\right)'}{I_n^\lambda F_c(g)(z)} \cdot h(z) + zh'(z) \\ &= H(z)h(z) + zh'(z). \end{aligned} \tag{30}$$

by using (24) and (30), we get

$$\frac{z\left(I_n^{\lambda} f(z)\right)'}{I_n^{\lambda} g(z)} = \frac{z(I_n^{\lambda} z F_c'(f)(z))' + cz(I_n^{\lambda} F_c(f)(z))'}{z\left(I_n^{\lambda} F_c(g)(z)\right)' + cI_n^{\lambda} F_c(g)(z)}$$

$$= \frac{\frac{z(I_n^{\lambda} z F_c'(f)(z)))'}{I_n^{\lambda} F_c(g)(z)} + c\frac{z(I_n^{\lambda} F_c(f)(z))'}{I_n^{\lambda} F_c(g)(z)}}{\frac{z\left(I_n^{\lambda} F_c(g)(z)\right)'}{I_n^{\lambda} F_c(g)(z)} + c}$$

$$= \frac{H(z)h(z) + zh'(z) + ch(z)}{H(z) + c}$$

$$= h(z) + \frac{zh'(z)}{H(z)+c} \prec \; q_{k,\gamma}(z). \qquad (31)$$

Letting $B(z) = \frac{1}{H(z)+\lambda}$ in (31), we get $\Re(B(z)) > 0$ if $c > \frac{-(k+\gamma)}{k+1}$.
Hence, applying Lemma 2 for A and B as described we can show that $h(z) \prec \; q_{k,\gamma}(z)$, so that $I_n^{\lambda} F_C(f)(z) \in UCC(k,\gamma,\beta)$.

Theorem 8. Put $c > \frac{-(k+\gamma)}{k+1}$.
If $I_n^{\lambda} f(z) \in UQC(k,\gamma,\beta)$, then $I_n^{\lambda} F_C(f)(z) \in UQC(k,\gamma,\beta)$.
Proof. The proof of Theorem 8, we can get it by using Theorem 7.

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