On Some Types of Strongly Connected Sets in Topological Spaces

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Abstract

The aim of this paper is to introduce and investigate some types of strongly connected sets in topological spaces, including strongly g-connected sets, strongly semi-connected sets, and strongly preconnected sets.

Keywords: Strongly connected set, Connected set, Semi-open set, g-open set, Preopen set.

1. Introduction and Preliminaries

The concept of strongly connected sets was introduced by Levine in [8], where he defined and investigated strong connectivity and strong local connectivity. Earlier, Levine in [6] and [7] introduced the concepts of generalized closed sets and semi-open sets in topological spaces. The concept of preopen sets was defined in [9]. Many topological concepts and properties related to connectedness, semi-open sets, generalized closed sets, and preopen sets have been defined and studied extensively by many topologists; see, for example, [1, 3, 5, 10, 12, 13].

The purpose of this paper is to introduce some new types of strongly connected sets in topological spaces using semi-open sets, generalized open sets, and preopen sets. We study several important properties and prove interesting results.

To state our theorems, we give some preliminary definitions. Throughout this paper, X and Y denote topological spaces.

Definition 1. [8] Let A be a subset of a topological space X. A is said to be strongly connected if $A \subseteq U_1$ or $A \subseteq U_2$ whenever $A \subseteq U_1 \cup U_2$, U_1, U_2 are open sets.

Remark 1. [8] A strongly connected subset of a topological space is connected.

Definition 2. A subset A of X is said to be:

(1) generalized closed (briefly, g-closed) [6] if $\bar{A} \subseteq U$ whenever U is open and $A \subseteq U$. The complement of a g-closed set is called generalized open (briefly, g-open).

(2) semi-open set [7] if $A \subseteq \overline{A^{\circ}}$, i.e., if there exists an open set V in X such that $V \subseteq A \subseteq \bar{V}$. The complement of a semi-open set is called semi-closed [4].
(3) preopen set [9] if $A \subseteq (\bar{A})^{\circ}$.
The collection of all semi-open sets, semi-closed sets, g-open sets, g-closed sets in X will be denoted by $SO(X), SF(X), GO(X), GF(X)$ respectively.
Definition 3. A function $f: X \to Y$ is said to be
(1) irresolute [5] (resp., g-irresolute [2], pre-irresolute [10]) if the inverse image of every semi-open (resp., g-open, preopen) set in Y is semi-open (resp., g-open, preopen) in X.
(2) g-continuous [2] if the inverse image of every open set in Y is g-open in X.

2. Strongly g-connected Sets in Topological Spaces

Definition 4. A subset A of X is said to be strongly g-connected if $A \subseteq U_1$ or $A \subseteq U_2$ whenever $A \subseteq U_1 \cup U_2$, U_1, U_2 are g-open sets.
Remark 2. A strongly g-connected set is both strongly connected and connected. The converse is not true, as shown by:
Example 1. Let $X = \{1,3,5,6\}$ and let $\tau = \{\phi, X, \{1,3\}, \{1,3,5\}, \{3\}, \{3,5\}\}$, so $GO(X) = \{\phi, X, \{1,3\}, \{1,3,5\}, \{3\}, \{3,5\}, \{1,5\}, \{5\}, \{1\}\}$. The set $A = \{1,3\}$ is both strongly connected and connected, but A is not strongly g-connected.
Remark 3. If A is strongly g-connected and $B \subseteq A$, then B need not be strongly g-connected, as shown in the following example.
Example 2. Let $X = \{2,4,6\}$ and let $\tau = \{\phi, X, \{2,4\}, \{2\}, \{4\}\}$, so $GO(X) = \tau$ and X is strongly g-connected. If $B = \{2,4\}$, then $B \subseteq X$ and B is not strongly g-connected.
Remark 4. If A and B are strongly g-connected subsets of X with $A \cap B \neq \phi$, then $A \cup B$ need not be strongly g-connected, as shown by:
Example 3. Let $X = \{1,3,4\}$ and let $\tau = \{\phi, X, \{1\}, \{1,3\}, \{1,4\}\}$, so $GO(X) = \tau$. Let $A = \{1,3\}$ and $B = \{1,4\}$, so $A \cap B \neq \phi$ and each of A and B is strongly g-connected. But $A \cup B = X$, which is not strongly g-connected.

Theorem 1. If $f: X \to Y$ is g-irresolute (resp., g-continuous) and E is a strongly g-connected set in X, then $f(E)$ is strongly g-connected (resp., strongly connected) in Y.

Proof. If $f(E)$ is not strongly g-connected in Y, then $f(E) \subseteq V_1 \cup V_2$ for some g-open sets V_1, V_2 and $f(E) \nsubseteq V_1$, $f(E) \nsubseteq V_2$. But then $E \subseteq f^{-1}(V_1) \cup f^{-1}(V_2)$ and $E \nsubseteq f^{-1}(V_1)$, $E \nsubseteq f^{-1}(V_2)$ where $f^{-1}(V_1)$, $f^{-1}(V_2)$ are g-open in X since f is g-irresolute. Therefore, E is not a strongly g-connected set in X. The proof of the second part follows along the same lines.

Corollary 1. Strong g-connectedness is preserved by g-irresolute surjections.

A space X is strongly g-connected if it cannot be expressed as the union of two proper g-open sets. Equivalently, X is strongly g-connected if any two g-closed sets in X intersect. From this definition, the proof of the following results follows immediately.

Theorem 2. If X is strongly g-connected then

(1) X has at most one g-closed singleton.

(2) X is not a T_1 space, so X is also not Hausdorff.

(3) X is normal space.

(4) The only subsets of X both g-closed and g-open are φ and X.

Remark 5. If $\tau_1 \subseteq \tau_2$ and if (X, τ_2) is strongly g-connected, then (X, τ_1) is not necessarily strongly g-connected. This is illustrated in the next example.

Example 4. Let $X = \{1,3,5\}$ and let τ_1 be the indiscrete topology, so $GO_1(X)$ is the power set of X. Let $\tau_2 = \{\phi, X, \{1\}, \{3\}, \{1,3\}\}$, so $GO_2(X) = \tau_2$. Now, $\tau_1 \subseteq \tau_2$ and (X, τ_2) is strongly g-connected but (X, τ_1) is not.

3. Strongly Semi-connected Sets

Definition 5. A subset A of X is called strongly semi-connected if $A \subseteq U_1$ or $A \subseteq U_2$ whenever $A \subseteq U_1 \cup U_2$, U_1, U_2 are semi-open sets.

Remark 6. Every strongly semi-connected set is both strongly connected and connected. The converse is false as shown by:

Example 5. Let $X = \{1,2,3,4\}$ and $\tau = \{\phi, X, \{1,2\}, \{1,2,3\}, \{2\}, \{3,2\}\}$, so $SO(X) = \{\phi, X, \{1,2\}, \{1,2,3\}, \{2\}, \{3,2\}, \{2,4\}, \{1,2,4\}, \{3,2,4\}\}$. If

$A = \{2,3,4\}$ then A is strongly connected and connected but not strongly semi-connected.

Remark 7. If A and B are strongly semi-connected in X and $A \cap B \neq \phi$, then $A \cup B$ need not be strongly semi-connected, as shown by the next example.

Example 6. Let $X = \{1,3,5,6\}$ and $\tau = \{\phi, X, \{5,6\}\}$, so we have $SO(X) = \{\phi, X, \{5,6\}, \{1,5,6\}, \{3,5,6\}\}$. If $A = \{1,5,6\}$, $B = \{3,6\}$, then A and B are strongly semi-connected and $A \cap B \neq \phi$; but $A \cup B$ is not strongly semi-connected.

The proof of the following theorem follows along the same lines as in Theorem 1.

Theorem 3. If $g: X \to Y$ is irresolute and E is a strongly semi-connected set in X, then $g(E)$ is strongly semi-connected in Y.

A space X is strongly semi-connected if it cannot be expressed as the union of two proper semi-open sets. Now, we have the following theorem.

Theorem 4. A space (X, τ) is strongly semi-connected if and only if either τ is the indiscrete topology or $\tau = \{\phi, X, X\backslash\{a\}\}$ for some $a \in X$.

Proof. If $|X| < 3$, the proof follows immediately; so suppose that $|X| \geq 3$. If $\tau = \{\phi, X\}$ then $SO(X) = \{\phi, X\}$ and if $\tau = \{\phi, X, X\backslash\{a\}\}$ for some $a \in X$, then $SO(X) = \{\phi, X, X\backslash\{a\}\}$; so in both cases, X is strongly semi-connected. Now, if the condition above does not hold, so $|\tau| \geq 3$ and $\tau \neq \{\phi, X, X\backslash\{a\}\}$, then there is a proper open subset U of X such that $U \neq X\backslash\{a\}$ for any $a \in X$. So we have two cases:

(1) If $\bar{U} \neq X$, then $\bar{U}$ is a semi-open proper subset of X since $U \subseteq \bar{U} \subseteq \bar{U}$. Since $\bar{U}^C$ is also semi-open and $X = \bar{U} \cup \bar{U}^C$, X is not a strongly semi-connected space.

(2) If $\bar{U} = X$, then for any $y \notin U$ we have $U \subseteq X\backslash\{y\} \subseteq X$, so $X\backslash\{y\}$ is semi-open. Therefore, $X = X\backslash\{a\} \cup X\backslash\{b\}$ for some $a, b \notin U$ and X is not strongly semi-connected.

Theorem 5. If A is a strongly semi-connected set in X, then the subspace (A, τ_A) is strongly semi-connected.

Proof. Suppose (A, τ_A) is not strongly semi-connected, then $A = W_1 \cup W_2$ for some proper semi-open sets $W_1, W_2 \in \tau_A$. But $W_1 = A \cap V_1$ and $W_2 = A \cap V_2$

for some semi-open sets V_1, V_2 in X, and $A \nsubseteq V_1$, $A \nsubseteq V_2$. Since $A \subseteq V_1 \cup V_2$, it follows that A is not a strongly semi-connected subset of X.

Corollary 2. If A is a strongly semi-connected set in X, then (A, τ_A) is either the indiscrete space or $\tau_A = \{\phi, A, A\backslash\{a\}\}$ for some $a \in A$.

Remark 8. The converse of Theorem 5 is not true in general, as illustrated by:

Example 7. Let $X = \{1,2,4,5\}$ and let $\tau = \{\phi, X, \{1,2\}\}$, so we have $SO(X) = \{\phi, X, \{1,2\}, \{1,2,4\}, \{1,2,5\}\}$. If $A = \{4,5\}$, then (A, τ_A) is the indiscrete space, so it is a strongly semi-connected space. But $A \subseteq \{1,2,4\} \cup \{1,2,5\}$ and each of $\{1,2,4\}$ and $\{1,2,5\}$ is semi-open, so A is not strongly semi-connected as a subset of X.

4. Strongly Preconnected Sets

Definition 6. Let A be a subset of X. A is strongly preconnected if it satisfies the condition: if $A \subseteq U_1 \cup U_2$ and U_1, U_2 are preopen sets in X then either $A \subseteq U_1$ or $A \subseteq U_2$.

Remark 9. Every strongly preconnected set is strongly connected and connected. The converse is false as shown by:

Example 8. Let $X = \{1,2,4,5\}$ and $\tau = \{\phi, X, \{1,2\}, \{1,2,4\}, \{2\}, \{4,2\}\}$. If $A = \{2,5,4\}$, then A is strongly connected and connected but not strongly preconnected, since $A \subseteq \{1,2,5\} \cup \{2,4\}$ and each of $\{1,2,5\}$ and $\{2,4\}$ is preopen set.

Remark 10. If A and B are strongly preconnected sets in X and $A \cap B \neq \phi$, then $A \cup B$ need not be strongly preconnected as shown by :

Example 9. Let $X = \{1,2,4,5\}$ and $\tau = \{\phi, X, \{4\}\}$. If $A = \{1,4\}$, $B = \{2,4\}$, then A and B are strongly preconnected and $A \cap B \neq \phi$. But $A \cup B = \{1,2,4\}$ which is not strongly preconnected, since there are preopen sets $\{2,4,5\}$ and $\{1,4\}$ with $A \cup B \subseteq \{2,4,5\} \cup \{1,4\}$.

Theorem 6. If $f: X \to Y$ is pre-irresolute, then the image of any strongly preconnected set in X is strongly preconnected in Y.

Proof. The proof follows in a similar way as in Theorem 1.

A space X is strongly preconnected if it cannot be written as the union of two proper preopen sets. Now, we have the following theorem.

Theorem 7. A space (X,τ) is strongly preconnected if and only if τ is the excluded point topology.

Proof. If τ is the excluded point topology, then the family of all preopen sets in X is τ. Obviously, X is strongly preconnected. Conversely, if τ is not the excluded point topology, then either X is the discrete space, which is not strongly preconnected, or there exist distinct points $a,b \in X$ such that $\{a\}$ and $\{b\}$ are not open. In this case, each of $X\backslash\{a\}$ and $X\backslash\{b\}$ is either open or dense. So, each of $X\backslash\{a\}$ and $X\backslash\{b\}$ is preopen set in X with $X\backslash\{a\} \cup X\backslash\{b\} = X$. Therefore, X is not strongly preconnected.

Remark 11. If $A \subseteq X$ and the subspace (A,τ_A) is strongly preconnected, then A need not be strongly preconnected as a subset of X. We illustrate this by the following example.

Example 10. Let $X = \{1,3,4,5,6\}$ and let τ be the topology on X defined as $\tau = \{\phi, X, \{3\}, \{1,3\}, \{5,6\}, \{3,5,6\}, \{1,3,5,6\}\}$. If $A = \{3,4,5\}$, then (A,τ_A) is the excluded point space so it is strongly preconnected. But $A \subseteq \{3,4,6\} \cup \{3,5\}$ and each of $\{3,4,6\}$ and $\{3,5\}$ is preopen subset of X. Hence, A is not strongly preconnected as a subset of X.

5. Conclusion

In this research, some new types of strongly connected sets were defined and studied using different types of open sets. Other forms of strong connectedness could be studied using other kinds of open sets, such as semi-g-open sets or semi-preopen sets.

References

[1] A. M. A. Ahmed, Semi-perfect Sets in Topological Spaces, *Journal of Humanitarian and Applied Sciences*, 11(2021), 298-301.

[2] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, *Mem. Fac. Sci. Kochi Univ. Ser. A. Math.,* 12(1991), 5-13.

[3] J. Cao, M. Ganster and I. Reilly, On generalized closed sets, *Topology & appl*, 123(2002), 37-46.

[4] S. G. Crossley and S. K. Hildebrand, Semi-closure, *Texas, J. Sci.,* 22(1971), 99 - 112.

[5] S. G. Crossley and S.K. Hildebr, Semi-topological properties, *Fund. Math*., 74(1972), 233-254.

[6] N. Levine, Generalized closed sets in topological spaces, *Rend. Circ. Mat. Palermo*., 19(1970), 89-96.

[7] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly.,* 70(1963), 36 – 41.

[8] N. Levine, Strongly connected sets in topology, *Amer. Math. Monthly.*, 72(1965),1098-1101.

[9] A. S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous mapping and weak precontinuous mapping, *Proc. Math. Phy. Soc. Egypt*, 53(1982), 47-53.

[10] I. L. Reilly and M.K. Vamanamurthy, On α-continuity in topological spaces, *Acta Math. Hungar*., 45(1-2)(1985), 27-32.

[11] T. Richmond, *General Topology*: An Introduction. *Berlin, Boston: De Gruyter*, 2020.

[12] D. Sasikala and M. Deepa, An elementary approach on hyperconnected spaces, *Turk. J. of Comput. Math. Educ*., 12(9)(2021), 946-950.

[13] S. Sekar and G. Kumar, On α-generalized regular-interior and α-generalized regular-closure in Topological Spaces, *Malaya J. Mat*., 5(1)(2017), 115-121.